1201
JAN 08

University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2008
Mathematics 1201
Friday 11 January 2008 1.30 – 3.30 or 2.15 – 4.15

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination*

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V. Explain what is meant by saying that
i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V ;
ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.

Suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V and that $\mathbf{v}_n$ can be expressed as a linear combination

$$\mathbf{v}_n = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_{n-1} \mathbf{v}_{n-1}.$$

Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ also spans V.

State the Exchange Lemma, and explain how it is used in formulating the idea of the *dimension* of a vector space.

In each case below, decide whether the given vectors are linearly independent over the field $\mathbf{Q}$ of rational numbers. If they are not, give an explicit dependence relation between them.

(a) $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}$

PLEASE TURN OVER

2. Let $A = (a_{ij})$ where $1 \le i \le m$ and $1 \le j \le n$, and $B = (b_{kl})$ where $1 \le k \le M$ and $1 \le l \le N$, be matrices over a field $\mathbf{F}$. State the condition under which the product AB exists and give the formal definition of AB in this case. The basic $m \times m$ matrix $\epsilon(r,s)$ is defined by

$$\epsilon(r,s)_{ij} = \delta_{ri}\delta_{sj}$$

where 'δ' is the Kronecker delta symbol. Prove that

$$\epsilon(r,s)\epsilon(s,t) = \epsilon(r,t)$$

and compute $\epsilon(r,s)\epsilon(u,t)$ when $s \neq u$.

Find A^{-1} when

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}.$$

Express both A^{-1} and A as products of elementary matrices.

3. Let V, W be vector spaces over a field $\mathbf{F}$; explain what is meant by a *linear mapping* $T : V \to W$, and define
(i) the kernel, $\mathrm{Ker}(T)$; (ii) the image, $\mathrm{Im}(T)$.

State and prove a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Let $T_A : \mathbf{Q}^6 \to \mathbf{Q}^4$ be the linear mapping $T_A(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 1 \\ -1 & -2 & -1 & -3 & 0 & -2 \\ 3 & 6 & 1 & 5 & 1 & 5 \\ 2 & 4 & -1 & 0 & 1 & 2 \end{pmatrix}.$$

Find (i) $\dim \mathrm{Ker}(T_A)$; (ii) a basis for $\mathrm{Ker}(T_A)$; (iii) a basis for $\mathrm{Im}(T_A)$.

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4. Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that
(i) f is injective ; (ii) f is surjective ; (iii) f is invertible.

In each case below decide, giving your explanation, whether the given mapping is (a) injective (b) surjective :

(i) $f : \mathbf{Z} \to \mathbf{Z}$; $f(x) = x^3 - 2x$;

(ii) $g : \mathbf{R} \to \mathbf{R}$; $g(x) = x^3 - 2x$.

Let $\mathcal{P}_8(\mathbf{R})$ be the vector space of polynomials of degree ≤ 8 over the field $\mathbf{R}$ and let

$$D : \mathcal{P}_8(\mathbf{R}) \to \mathcal{P}_8(\mathbf{R})$$

be the linear map given by differentiation. Write down the least integer n for which $D^n = 0$ on $\mathcal{P}_8(\mathbf{R})$.
By factorisation of the formal expression $D^n + \mathrm{I}$, or otherwise, show that the mapping

$$D^6 - D^3 + \mathrm{I} : \mathcal{P}_8(\mathbf{R}) \to \mathcal{P}_8(\mathbf{R})$$

is invertible, and write down an expression for its inverse in terms of D.

Hence or otherwise write down the unique solution $\alpha \in \mathcal{P}_8(\mathbf{R})$ to the differential equation

$$\frac{d^6\alpha}{dx^6} - \frac{d^3\alpha}{dx^3} + \alpha = x^4 + x.$$

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5. (i) Replace the formula

$$(p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$$

by an equivalent formula which *does not contain* either $\vee$ or $\neg$ but which *does contain* $\wedge$ and $\Longrightarrow$.

(ii) Negate the following formula, and replace it by an equivalent formula which *does not involve* either $\neg$, $\vee$ or $\wedge$:

$$\neg(\forall x)(\exists y)(\neg P(x,y) \wedge Q(x,y)) \ \bigwedge \ (\exists x)(\forall y)(\neg Q(x,y) \wedge R(x,y)).$$

(iii) Define sign(σ) when σ is a permutation of the set $\{1, \ldots, n\}$. Explain what is meant by an *adjacent transposition* .

If τ is a transposition show that τ is a product of adjacent transpositions, and so deduce that

$$\text{sign}(\tau) = -1.$$

Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 3 & 11 & 13 & 12 & 14 & 4 & 5 & 7 & 6 & 2 & 8 & 15 & 1 & 9 \end{pmatrix}$$

into a product of disjoint cycles and hence compute

(a) sign(σ) and (b) ord(σ).

CONTINUED

6. Let $T : U \to V$ be a linear map between vector spaces U , V and let $\mathcal{E} = (e_i)_{1 \le i \le n}$ be a basis for U and $\Phi = (\varphi_i)_{1 \le i \le p}$ be a basis for V. Explain what is meant by the matrix $m(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) and Φ (on the right) and show that

$$m(\mathrm{Id}_U)^{\mathcal{E}}_{\mathcal{E}} = \mathrm{I}_n.$$

If $S : V \to W$ is also linear and $\Psi = (\psi_k)_{1 \le k \le q}$ is a basis for W prove that

$$m(S \circ T)^{\Psi}_{\mathcal{E}} = m(S)^{\Psi}_{\Phi} m(T)^{\Phi}_{\mathcal{E}}.$$

Let $T : \mathbf{F}^2 \to \mathbf{F}^2$ be the mapping $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 + x_2 \\ -4x_1 + 3x_2 \end{pmatrix}$

and let $\mathcal{E}$, Φ be the bases $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$; $\Phi = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$.

Write down (i) $m(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $m(\mathrm{Id})^{\mathcal{E}}_{\Phi}$.

Hence find $m(T)^{\Phi}_{\Phi}$.

END OF PAPER